Math 246A Lecture 26 Notes

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1 Normal Families of Meromorphic Functions and Marty's Theorem

1.1 Normal families of meromorphic functions

Recall that if $z, w \in \mathbb{C}$, the spherical distance is

$$d_S(z,w) = \frac{|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}},$$

$$d_S(z,\infty) = \frac{2}{\sqrt{1+|z|^2}}.$$

This is the distance between the points viewed as points on the Riemann sphere, $\mathbb{C}^*$. It has the property that

$$d_S(1/z, 1/w) = d_S(z,w).$$

Recall the definition of meromorphic functions $f : \mathbb{C}^* \to \mathbb{C}^*$. This means that there exists $\{p_n\}$, a discrete closed subset of Ω such that $f \in H(\Omega \setminus \{p_1, \dots, p_n\})$, so

$$f(z) = \frac{g(z)}{(z-p_n)^{k_n}}$$

in a small punctured neighborhood around each p_n.

Definition 1.1. $\mathcal{F}$ is **normal in the classical sense** if whenever $\{f_n\}$ is a sequence in $\mathcal{F}$, there exists a subsequence f_{n_j} and $f : \Omega \to \mathbb{C}^*$ such that for all compact $K \subseteq \Omega$, $\sup_K d_S(f_{n_j}(z), f(z)) \to 0$

Basically, we allow the functions to be ∞, as well.

Remark 1.1. f is continuous on Ω.

Remark 1.2. If $\Omega \subseteq \mathbb{C}$ and $\mathcal{F} \subseteq H(\Omega)$, then $\mathcal{F}$ is normal iff $\mathcal{F}$ is normal in the classical sense.

Lemma 1.1. *If $\{f_n\}$ is meromorphic on Ω, $\sup_K d_S(f_n(z), f(z)) \to 0$ for all compact $K \subseteq \Omega$, then f is meromorphic on Ω or $f = \infty$.*

Proof. Suppose $f \neq \infty$. Then there exists some $p \in \Omega$ such that $f(p) \neq \infty$. Then $d_S(f(p), \infty) > 2\varepsilon > 0$. Let $U = \{z : d_S(z, p) < \delta\} \subseteq \Omega$ be such that $\overline{U} \subseteq \Omega$, and take δ small enough and n large enough so that $n > N \implies \sup_{\overline{U}} d_S(f_n(z), f_n(p)) < \varepsilon$; this exists is by Arzelaá-Ascoli for maps $\Omega \to \mathbb{R}^3$. Then $f_n \in H(U)$ for large enough n, so $f \in H(U)$. So we get that $f \in H(\Omega \setminus f^{-1}(\{\infty\}))$, since $f^{-1}(\{\infty\})$ is a closed subset of Ω. Also, $1/f_n \to 1/f$ uniformly on compact subsets. since $d_S(1/z, 1/w) = d(z, w)$. So $1/f \in H(\Omega \setminus \{f^{-1}(\{0\})\}$. Therefore, f has a pole at each a such that $f(a) = \infty$. $\square$

1.2 Spherical derivative, spherical length, and Marty's theorem

Definition 1.2. The "**spherical derivarive**" of f at z is

$$\rho(f)(z) = \frac{2|f'(z)|}{1 + |f(z)|^2}.$$

Observe that

$$\rho(1/f)(z) = \frac{2\frac{|f'(z)|}{|f(z)|^2}}{1 + \frac{1}{|f(z)|^2}} = \rho(f)(z).$$

Also note that

$$\lim_{w \to z} \frac{d_s(f(z), f(w))}{|z - w|} = \lim_{w \to z} \frac{\frac{|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}}{|z - w|} = \rho(f)(z).$$

Definition 1.3. Let $z, w \in \Omega \setminus \{\infty\}$ and let $[z, w] = \{tw + (1 - t)z : 0 \leq t \leq 1\}$. Assume $[z, w] \subseteq \Omega$. Then the **spherical length** of $f([z, w])$ is

$$\sup \left\{ \sum_{j=1}^{n} d_S(f(z_{j-1}), f(z_j)) : z_j = t_j w + (1 - t_j)z, 0 = t_0 < t_1 < \cdots < t_n = 1 \right\}.$$

This is approximately

$$\sum_{j=1}^{n} \rho(f)(z_j)|z_j - z_{j-1}| \xrightarrow{n \to \infty} \int_0^1 \rho(f)(z(t))|z'(t)|\, dt.$$

So if $\rho(f) \leq M$,

$$d_S(f(z), f(w)) \leq M|z - w|.$$

This gives us the following theorem.

Theorem 1.1 (Marty)**.** *Let $\mathcal{F}$ be a family or meromorphic functions on a domain $\Omega \subseteq \mathbb{C}^*$. Then $\mathcal{F}$ is normal iff for all compact $K \subseteq \Omega$,*

$$\sup_{f \in \mathcal{F}} \sup_{z \in \Omega} \rho(f)(z) < \infty.$$